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The splitting of separatrices, the branching of solutions and non-integrability in the problem of the motion of a spherical pendulum with an oscillating suspension point☆

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Abstract

The effectiveness of the results obtained previously in [Dovbysh SA. Transversal intersection of separatrices and non-existence of an analytical integral in multidimensional systems. In: Ambrosetti A, Dell Antonio GF, editors. *Variational and Local Methods in the Study of Hamiltonian Systems*. Singapore, etc: World Scientific; 1995. p. 156–65; Dovbysh SA. Transversal intersection of separatrices, the structure of a set of quasi-random motions and the non-existence of an analytic integral in multidimensional systems. *Uspekhi Mat Nauk* 1996; **51**(4): 153–54; Dovbysh SA. Transversal intersection of separatrices and branching of solutions as obstructions to the existence of an analytic integral in many-dimensional systems. I. Basic result: Separatrices of hyperbolic periodic points. *Collect Math* 1999; **50**(2): 119–97; Dovbysh SA. Branching of the solutions in the complex domain from the point of view of symbolic dynamics and the non-integrability of multidimensional systems. *Dokl Ross Akad Nauk* 1998; **361**(3): 303–6] on the non-integrability of multidimensional systems is illustrated using the example of the problem of the motion of a spherical pendulum with a suspension point performing small periodic oscillations. With this aim, the splitting of the separatrices of the unstable equilibrium position and the branching of the solutions are investigated. It is shown that the separatrices are split for any law of motion of the suspension point, and a simple criterion of the presence of their transversal intersection is obtained. The validity of the non-integrability result, based on a combination of the conditions related to the splitting of multidimensional separatrices and to the branching of the solutions, is also pointed out.

Essentially new conditions, which guarantee the non-integrability of multidimensional dynamical systems in the strongest analytical sense, i.e. the absence of a non-constant analytic (and even meromorphic) integral at a level of a priori first integrals were obtained earlier by the author. These conditions are related to the transversal intersection of multidimensional invariant manifolds of hyperbolic periodic solutions (separatrices)[1–3] or to the branching of solutions at the complex region.[4] Corresponding to these two cases, it was established that there was no integral meromorphic in a certain real or complex region. Moreover, there is no analytical or even meromorphic vector field, which commutes with the vector field of the phase flow (generating the local symmetry of the system) and which is not obtained from the latter by multiplication by a certain constant.

For a plane pendulum with a suspension point performing vertical periodic oscillations, it was found in Ref. 5,6 that in this Hamiltonian system with one and a half degrees of freedom, two-dimensional separatrices are split with an

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intersection for arbitrary periodic motion of the suspension point (errors were made when calculating the corresponding Mel'nikov integrals in Ref. 5,6; however, the results obtained there are valid, which can be seen from the correct formulae presented below). This fact implies the analytical integrability, i.e. the non-existence of an analytic first integral in the three-dimensional extended phase space (the instants of time, which are distinguished in the oscillation period of the suspension point, are identified; hence the first integrals which have this period in the time variable are considered). When investigating the splitting of the separatrices in the case of a plane pendulum with a suspension point performing small horizontal sinusoidal oscillations, the splitting and transversal intersection of the separatrices was found,[7] which implies the analytical non-integrability.

The splitting of separatrices for a spherical damped and magnetized pendulum with a vibrating suspension point, which interacts with a fixed magnet repelling the pendulum from its slower equilibrium position, which therefore becomes unstable (if the repulsive force exceeds the restoring force) was considered in Ref. 8,9. The case where the damping is asymmetric and the suspension point performs horizontal sinusoidal oscillations was considered. To investigate the system, only terms of no higher than the third order were retained in the equations of motion; this can be justified when the separatrices are confined in a small neighbourhood of the lower equilibrium position. This condition is satisfied if the power of the repulsing magnetic is confined to a certain fairly narrow interval; the corresponding limitations were not discussed.

In this paper a complete and rigorous investigation of the splitting of the separatrices is carried out, and results on non-integrability are obtained for the first time, in the problem of the motion of a spherical pendulum with a vibrating suspension point.

1. Formulation of the problem

A spherical pendulum is a point mass M forced to remain at a constant distance l from a fixed suspension point S. It is assumed that this constraint is ideal and the only external force is the uniform force of gravity. This system is an autonomous Hamiltonian system with two degrees of freedom and with the first integrals of the energy and areas. There is a single unstable equilibrium position O such that the vector $\rightarrow SO$ is directed vertically upwards. This equilibrium position is hyperbolic, and all the solutions, which asymptotically approach the point O when the time approaches $-\infty$ or $+\infty$, constitute an unstable manifold (separatrice) W^- and a stable manifold (separatrice) W^+ in the 4-dimensional phase space. In the system under consideration these two-dimensional separatrices coincide (are doubled), since they form a common level of the two first integrals.

Suppose the suspension point S is forced to perform small periodic oscillations, which corresponds to a small time-periodic perturbation of the original autonomous system. Then the hyperbolic equilibrium position and its separatrices are perturbed in the hyperbolic periodic solution and the corresponding three-dimensional separatrices in the 5-dimensional extended phase space. The perturbation of the doubled separatrices leads, generally speaking, to their splitting. This phenomenon of the "splitting of the separatrices" is well known and can be investigated using Mel'nikov's method, which is briefly discussed below.

It is convenient to use dimensionless variables, in which the mass of the point M, the length of the pendulum l and the acceleration due to gravity g are taken to be equal to unity. To investigate the perturbed spherical pendulum, we will introduce a right Cartesian system of coordinates (x, y, z) with origin at the suspension point S and with its z axis directed vertically upwards, and we will denote the vector $\rightarrow SM$, drawn from the suspension point to the point M, by $\gamma = (\gamma_x, \gamma_y, \gamma_z)$. The configuration space of the system is the two-dimensional unit sphere $S^2 = \{\gamma\colon |\gamma| = 1\}$, while the phase space is the tangent bundle TS^2 over the sphere, i.e. a set of pairs of vectors of the coordinates and velocities $w = (\gamma, \dot{\gamma})$, where $\gamma \in S^2$ and $\dot{\gamma} \perp \gamma$. The point q of the phase space, corresponding to the upper unstable equilibrium position O, is defined by the equality $q = (n, 0)$, where $n = (0, 0, 1)$. The first integrals of the energy and areas have the form

$$E = \frac{1}{2}(\dot{\boldsymbol{\gamma}} \cdot \dot{\boldsymbol{\gamma}}) + \boldsymbol{\gamma}_z, \quad h = (\boldsymbol{\gamma} \times \dot{\boldsymbol{\gamma}})_z = \gamma_1\dot{\gamma}_2 - \gamma_2\dot{\gamma}_1$$

Further, the effect of the motion of the suspension point S can be represented as the action of a corresponding inertia force. Hence, the investigation of the system where the point S is forced to move with an acceleration $\mathbf{a}(t)$ is equivalent to an investigation of a spherical pendulum with a fixed suspension point and with an additional external force $\mathbf{F} = -\mathbf{a}(t)$. In order to consider small oscillations of the point S, we will introduce a small parameter ε and replace $\mathbf{a}(t)$ by $\varepsilon\mathbf{a}(t)$.

2. The non-integrability conditions based on transversal intersection of the separatrices in multidimensional systems

We will recall the result on the non-integrability of multidimensional systems in its simplest form,[2] which relates to the case when there is a single hyperbolic periodic solution and a certain number of homoclinic trajectories. Moreover, we are here considering the case of a "narrow" spectrum, which is sufficient for application to a perturbed spherical pendulum, but enables the formulation to be simplified somewhat. In fact, for this case when formulating the result on non-integrability, certain geometrical conditions, relating to the mutual position of the tangent subspaces with respect to the separatrices at their points of intersection, automatically vanish (see Ref. 2 for more detail).

Definition 1. We will say that a non-degenerate linear operator $T:\mathbb{C}^n\to\mathbb{C}^n$ and a set $K\subset\mathbb{C}^n$ are in general position, if for any eigenvalue λ of the operator T the linear hull (over $\mathbb{C}$) of the union of the set K and the image of the operator $T-\lambda\cdot\mathrm{id}$ coincides with the whole ambient space $\mathbb{C}^n$.

We will give an equivalent form of Definition 1.

Definition 1′. The non-degenerate linear operator $T:\mathbb{C}^n\to\mathbb{C}^n$ and the set $K\subset\mathbb{C}^n$ are in general position if the linear hull of the set $\bigcup_{k=0}^{n-1}T^k(K)$ coincides with the whole ambient space $\mathbb{C}^n$.

This condition of the generality of the position admits of a simple description in terms of the Jordan form of the operator T and of the corresponding coordinates of the points of the set K, and also persists under small perturbations of the operator T and the set K.[1,3] We note the following as an important corollary.

Remark 1. We will denote by V_r the maximum generalized invariant subspaces of T, corresponding to a certain partition of the spectrum of T into non-intersecting classes (in particular, these subspaces may correspond to all distinct eigenvalues) and let $T=\oplus_r T_r$ be the decomposition of T into linear operators T_r: $V_r\to V_r$. We will also denote by π_r the natural projection $\mathbb{C}^n\to V_r$ along V_r, $r'\neq r$. Then T and K are in general position if and only if T_r and $K_r=\pi_r(K)$ are in general position for each r.

Suppose q is a hyperbolic fixed point of a C^N-diffeomorphism S of an n-dimensional manifold M into itself; W^- and W^+ are its outgoing and incoming invariant manifold (separatrices), which, as is well known, are also manifolds of class C^N. Suppose the dimensions $W^\pm$ are equal to $n^\pm$ respectively (then $n^++n^-=n$) and r_m are certain transversal homoclinic points, so that at each point r_m the manifolds W^- and W^+ intersect transversely. We will say that the orbits of the points and q and r_m form a homoclinic structure.

Suppose λ_i $(1\le i\le n^+)$, μ_i $(1\le i\le n^-)$ are all the eigenvalues of the mapping S at the point q, where $0<|\lambda_i|<1<|\mu_i|$. We will assume, for simplicity, that both parts of the spectrum lying inside and outside the unit circle are narrow in the sense that

$$\max_i|\lambda_i|^2<\min_i|\lambda_i| \quad\text{and}\quad \min_i|\mu_i|^2>\max_i|\mu_i| \tag{2.1}$$

Then N may be any integer number such that $N\ge 2$. By Sternberg's theorem[10] "linearizing" coordinates $y^\pm\in\mathbb{R}^{n^\pm}$ of the class C^N on $W^\pm$ exist, in which the mapping $S|W^\pm$ takes the linear form $y^\pm\mapsto J^\pm y^\pm$. For each of the two indices $\pm$ we will denote the set of $y^\pm$ -coordinates of the points $r_m\in W^\pm$ by $K^\pm\subset\mathbb{R}^{n^\pm}$ and we will assume that the linear mapping $J^\pm$ and the set $K^\pm$ are in general position.

Theorem 1. *When the conditions formulated above are satisfied, the diffeomorphism S has no non-constant real-meromorphic first integral in any neighbourhood of the homoclinic structure considered.*

If the diffeomorphism S is analytic, its separatrices W^- and W^+ and the linearizing coordinates on them are also analytic. Moreover, we note that these considerations transfer without any change to the complex case when S is a complex-analytic diffeomorphism. Then the separatrices $W^\pm$ and the linearizing coordinates $y^\pm\in\mathbb{C}^{n^\pm}$ on $W^\pm$ are also complex-analytic and a complete analogue of Theorem 1 occurs, which asserts the absence of a non-constant meromorphic first integral in the neighbourhood of the complex homoclinic structure considered.

The diffeomorphism S often arises in applications as a first return map (the Poincaré mapping) of the phase flow of a dynamical system, while a fixed hyperbolic point q of diffeomorphism S corresponds to a hyperbolic periodic trajectory of this system. Then the non-existence of an analytic or meromorphic integral of the original system is equivalent to the non-existence of such an integral for the Poincaré mapping.

Here, when considering the complex case, the complex-analytic linearizing coordinates are not extended over the whole space $\mathbb{C}^{n^\pm}$, and, generally speaking, are only determined in the neighbourhood of the point $y^\pm = 0$, unlike the real case. The reason for this is that the solutions are not extended over the whole region of the complex time variable. However, this fact is not essential for the complex analogue of Theorem 1 to be correct.

Remark 2. Theorem 1 can easily be used to prove the non-integrability of systems which are perturbed integrable systems. We will assume, for simplicity, that the diffeomorphisms considered are analytic. Suppose S_ε is a diffeomorphism which depends on a small parameter ε. We will assume that the "unperturbed" diffeomorphism S_0 has a fixed hyperbolic point q_0, such that the spectrum S_0 in q_0 satisfies inequalities (2.1). Then, for small perturbations of the mapping S_0, its fixed hyperbolic point q_0 and the separatrices $W_0^\pm$ will be only slightly perturbed, and inequalities (2.1) for the spectrum remain in force. Suppose the "unperturbed" diffeomorphism S_0 is in a certain sense integrable, while for all small $\varepsilon \neq 0$ there appear points $r_m(\varepsilon)$ of the transversal intersection of the perturbed separatrices W_ε^+ and W_ε^-, such that $r_m(\varepsilon) \to r_m$ as $\varepsilon \to 0$, where $r_m \in W_0^+ \cap W_0^-$ (one can detect transversal homoclinic points arising under perturbation by using some version of the Mel'nikov method, developed both for phase flows described by ordinary differential equations and for diffeomorphisms; in Section 4 we will use the simplest multidimensional version of the method for phase flows). Finally, we will assume that for the mapping S_0 and the points r_m on its separatrices $W_0^\pm$ the conditions of the generality of position proposed in Theorem 1 are satisfied (we emphasise that doubly asymptotic points r_m cannot be transversal homoclinic points). Then, the conditions of Theorem 1 will be satisfied for the perturbed diffeomorphism S_ε and for the tranversal homoclinic points $r_m(\varepsilon)$ for all small $\varepsilon \neq 0$. Indeed, the required result follows directly from the following two facts: (1) under small perturbations of the mapping S_0, its separatrices $W_0^\pm$ and the linearizing coordinates $y^\pm$ on $W_0^\pm$ will be only slightly perturbed (see Ref. 3), and (2) as was pointed out above, the condition for the mapping $J^\pm$ and the set $K^\pm$ to be in general position is preserved when they are slightly perturbed.

3. The conditions of non-integrability based on branching of the solutions in multidimensional systems

We recall one of the non-integrability theorems.[4] For simplicity we will formulate a "narrow" spectrum in a somewhat simplified version. Consider a system of analytic ordinary differential equations

$$dx/dt = X(x, t) \tag{3.1}$$

in the complex domain of the independent variable t and phase variables x. In the case under discussion, the system will be τ-periodic in t. Therefore it is convenient to assume that the independent variable t ranges over a complex cylinder $\mathbb{C}/\tau\mathbb{Z}$. Suppose the extended phase variables (x, t) vary in the domain $D \subset \mathbb{C}^n \times (\mathbb{C}/\tau\mathbb{Z})$, where the right-hand side of system (3.1) has the form

$$X(x, t) = X_0(x) + \varepsilon X_1(x, t) + O(\varepsilon^2) \tag{3.2}$$

i.e. it is a perturbation of an autonomous system. Moreover, suppose $X_0(q) = 0$ for a certain point $q \in \mathbb{C}^n$ and $\Gamma = \Gamma_0 : [0, 1] \to \mathbb{C}\{t\}$ is a certain contour such that $\Gamma(1) - \Gamma(0) = \tau$ and $\{q\} \times \Gamma \subset D$; we will also introduce a domain

$$V = \{t: (q, t) \in D\} \supset \Gamma \equiv \Gamma_0 \text{ in } \mathbb{C}$$

Definition 2. We will say that the operator $T : \mathbb{C}^n \to \mathbb{C}^n$ and the $\mathbb{C}^n$-valued function f, holomorphic everywhere in V, apart from a finite collection of singularities, is in a general position if T and $K \subset \mathbb{C}^n$ are in general position in the sense of Definitions 1and 1′ where K is a set of residues of the function f in V.

We will choose the closed contours $\Gamma_k \subset V$ $(k \neq 0)$, passing through the point $t_0 = \Gamma_0(0)$ so that (1) each of them is contractible in the region V and surrounds exactly one singularity $t_k \in V$ of the function f, and (2) theoperator T and the subset

$$K(\{\Gamma_k\}) = \{\mathrm{Res}_{t_k} f\} \subseteq K \quad (k \neq 0)$$

are in general position.

Theorem 2. *Suppose the conditions formulated above are satisfied. We will denote by $\Lambda = (\partial X_0/\partial x)|_{x=q}$ the matrix of the unperturbed system (for $\varepsilon = 0$), linearized about the constant solution $x \equiv q$. Suppose the operator*

$$J = \exp(\tau\Lambda) \tag{3.3}$$

is hyperbolic (and so the unperturbed constant τ-periodic solution $x \equiv q$ is hyperbolic), and moreover, its eigenvalues λ_i, μ_i, which lie inside and outside the unit circle ($0 < |\lambda_i| < 1 < |\mu_i|$), satisfy conditions (2.1). Then, if the operator (3.3) and the function

$$f(t) = \exp(-\Lambda t) X_1(q, t) \tag{3.4}$$

are in general position, the system has no non-constant meromorphic first integral in any neighbourhood of the union of countours

$$\{q\} \times \cup_k \Gamma_k \subset D \tag{3.5}$$

for all sufficiently small $\varepsilon \neq 0$.

Remark 3. Suppose L^+ and L^- are invariant subspaces of the operator J, corresponding to the parts of the spectrum lying inside and outside the unit circle, and $J = J^+ \oplus J^-$ is the corresponding decomposition into the direct sum of the operators $J^\pm$: $L^\pm \to L^\pm$. Obviously L^+ and L^- are invariant subspaces of the operator $\tau\Lambda$, corresponding to the parts of the spectrum lying in the complex plane to the left and right of the real axis, and $J^\pm = \exp(\tau\Lambda^\pm)$, where $\Lambda = \Lambda^+ \oplus \Lambda^-$ is the decomposition into the direct sum of the operators $\Lambda^\pm$: $L^\pm \to L^\pm$. We will define the $L^\pm$-valued function

$$f^\pm(t) = \pi^\pm f(t) = \exp(-\Lambda^\pm t)\pi^\pm X_1(q, t)$$

where $\pi^\pm : \mathbb{C}^n \to L^\pm$ is the natural projection along $L^\pm$. According to Remark 1 the condition for the operator J and the function $f(t)$ to be in general position, is equivalent to the requirement that for each of the two indices $\pm$ the operator $J^\pm$ and the function $f^\pm(t)$ are in general position.

4. Combination of the non-integrability conditions based on the transversal intersection of the separatrices and on the branching of the solutions

We will formulate the result which directly includes Theorem 1 (for the case described in Remark 2, where the diffeomorphism S is the first return map of the periodic system close to an autonomous system) and Theorem 2.

Following Section 3, we will consider τ-periodic system (3.1) with right-hand side (3.2) such that $X_0(q) = 0$. Suppose $\Lambda = (\partial X_0/\partial x)|_{x=q}$ and the operator (3.3) satisfies the conditions of Theorem 2. We will denote by $W^\pm$ the complex separatrices of the hyperbolic point q of the system of equations $dx/dt = \tau X_0(x)$. Then $\tilde{W}_0^\pm = W_0^\pm \times (\mathbb{C}/\tau\mathbb{Z})$ are the separatrices of the hyperbolic periodic solution $x \equiv q$ of the unperturbed τ-periodic system $dx/dt = X_0(x)$. We will assume that, for all small $\varepsilon \neq 0$ there are lines $\gamma_m(\varepsilon)$ of transversal intersection of the perturbed separatrices $\tilde{W}_\varepsilon^\pm$, where

$$\gamma_m(\varepsilon) \to \gamma_m \quad \text{as} \quad \varepsilon \to 0; \quad \gamma_m \in \tilde{W}_0^+ \cap \tilde{W}_0^-$$

By virtue of a version of Sternberg's theorem, relating to flows, we can introduce linearizing coordinates $y^\pm$ on the separatrices $W_0^\pm$, in which the unperturbed system takes a linear form. Note that the subspaces $L^\pm$ coincide with the tangent subspaces to the separatrices $W_0^\pm$ at the point q. Hence, we can introduce linearizing coordinates $y^\pm$ on $W_0^\pm$, which range over $L^\pm = T_q W_0^\pm$ and are such that the differential of the corresponding mapping $\chi^\pm : L^\pm \to W_0^\pm$ (which puts the set of coordinates in correspondence with a point of the separatrix) at zero $0 \in L^\pm$ will be the identity mapping. Then, in these coordinates the restriction of the unperturbed system to its separatrix $W_0^\pm$ takes the form $dy^\pm/dt = \Lambda^\pm y^\pm$. In $(y^\pm, t)$ coordinates on $\tilde{W}_0^\pm$, the solution γ_m of the unperturbed system is specified by the formula $y^\pm = \exp(\Lambda^\pm t) z_m^\pm$, where $z_m^\pm \in L^\pm$ is a certain vector.

Further, suppose the contour $\Gamma_0 = \Gamma$ and the domain V are defined as in Section 3 and suppose the $\mathbb{C}^n$-valued function (3.4) is everywhere holomorphic in V, apart from a finite collection of singularities. We will denote by $K_1^\pm$ the set of all the elements $z_m^\pm$ and let $K_2^\pm = \pi^\pm(K_2)$, where K_2 is the set of residues of the function f in V. Obviously $K_2^\pm$ is the set of residues of the function $f^\pm$ in V. Suppose, finally, for each of the two indices $\pm$ the operator $J^\pm$ and the set $K^\pm = K_1^\pm \cup K_2^\pm \subset L^\pm$ are in general position.

Theorem 3. *If the conditions formulated above are satisfied, the system has no non-constant meromorphic integral in any neighbourhood of the closed set*

$$\bigcup_m \gamma_m \cup \left(\{q\} \times \bigcup_k \Gamma_k\right) \subset D$$

for all sufficiently small $\varepsilon \neq 0$.

The proofs of both Theorems 1 and 2 use quite related ideas and are based on a description by methods of symbolic dynamics for the set of trajectories which lie in fairly small neighbourhoods of the homoclinic structure or of the union of contours (3.5). The proof of Theorem 3 follows from the observation that the schemes of the proofs of Theorems 1 and 2 can be combined into one, thereby obtaining the combined non-integrability conditions described above.

5. Splitting of the separatrices of the unstable equilibrium position of the perturbed spherical pendulum

To investigate the splitting of the separatrices we will use Mel'nikov's method in its multidimensional version. Among the different known versions of the method, the most convenient one here relates to the perturbation of integrable systems and is based on consideration of the so-called Mel'nikov integrals (see Ref. 11). Note that henceforth we will use the non-Hamiltonian version of the method (the method itself in its Hamiltonian version and in a somewhat different form in fact already occurs in Poincaré's paper[12]).

In the problem of the motion of a plane pendulum, the doubly asymptotic solutions are given by the well-known formulae

$$\gamma_x = \pm\gamma_\perp(t'), \quad \gamma_\perp(t') = \frac{2\,\mathrm{sh}\,t'}{\mathrm{ch}^2 t'}, \quad \gamma_z = \gamma_\parallel(t') = 1 - \frac{2}{\mathrm{ch}^2 t'}, \quad t' = t_0 + t$$

(when $\gamma_y = 0$, i.e. the motion is performed in the xz plane). The corresponding solutions for the spherical pendulum $\gamma = \gamma^*(t + t_0, \theta)$ in the configuration space are obtained by the rotations around the vertical z axis and are given by the formulae

$$\boldsymbol{\gamma}^*(t, \theta): \quad \gamma_x = \gamma_\perp(t)\cos\theta, \quad \gamma_y = \gamma_\perp(t)\sin\theta, \quad \gamma_z = \gamma_\parallel(t)$$

where $\theta \bmod 2\pi$ is the rotation angle, which numbers the solutions, and t_0 is a parameter corresponding to the time shift of the solution. We will denote by $w^*(t, \theta) = (\gamma^*(t, \theta), \dot{\gamma}^*(t, \theta))$ the corresponding solution in the phase space. In the first order of perturbation theory, the mutual position of the perturbed separatrices is characterized by the two-dimensional Mel'nikov vector (function) $M = (M_E(t_0, \theta), M_j(t_0, \theta))$, the components of which specify the displacements of the separatrices in the directions of the two unperturbed integrals, respectively.

The exact meaning of this assertion is as follows. We denote by U a narrow neighbourhood of the unperturbed three-dimensional separatrix $\tilde{W}^\pm$ in the extended 5-dimensional phase space, which is not intersected with a small neighbourhood of the unperturbed periodic solution $w = q$, corresponding to the upper unstable equilibrium position O.

The doubled unperturbed separatrix is filled by the solutions $w = w^*(t + t_0, \theta)$, and we can therefore take (t, t_0, θ) as the coordinates on $\tilde{W}^\pm \cap U$. We can choose the unperturbed integrals E and j as the coordinates, transversal to the separatrix. In the system of coordinates (t, t_0, θ, E, j) obtained in the neighbourhood U, the perturbed separatrices $\tilde{W}^\pm$ are given by the equations

$$E = E^\pm(t, t_0, \theta) = O(\varepsilon), \quad j = j^\pm(t, t_0, \theta) = O(\varepsilon)$$

The displacements of the separatrices $\tilde{W}^+$ and $\tilde{W}^-$ in the directions of the coordinates E and j are functions that can be expanded in series in the small parameter ε

$$\Delta E = E^+(t, t_0, \theta) - E^-(t, t_0, \theta) = -\varepsilon M_E(t_0 - t, \theta) + \dots$$

$$\Delta j = j^+(t, t_0, \theta) - j^-(t, t_0, \theta) = -\varepsilon M_j(t_0 - t, \theta) + \dots$$

in which the coefficients of first-order terms are the components of Mel'nikov's vector. Hence, a transversal homoclinic solution of the perturbed system, $O(\varepsilon)$-close to the solution $w = w^*(t + t_0, \theta)$ of the unperturbed system, corresponds to each simple zero (t_0, θ) of Mel'nikov's vector (where the corresponding Jacobi matrix is non-degenerate). Note that Mel'nikov's vector has a period in the time variable t_0, equal to the period of the oscillations of the suspension point.

To calculate Mel'nikov's vector we note that the changes in the unperturbed integrals are given by the expressions

$$\dot{E} = (\dot{\boldsymbol{\gamma}} \cdot \mathbf{F}) = \varepsilon g_E, \quad g_E = -(\dot{\gamma}_x a_x + \dot{\gamma}_y a_y + \dot{\gamma}_z a_z)$$

$$(j)^{\boldsymbol{\cdot}} = (\boldsymbol{\gamma} \times \mathbf{F})_z = \varepsilon g_j, \quad g_j = \gamma_y a_x - \gamma_x a_y$$

Since the functions g_E and g_j vanish on the unperturbed periodic solution, the formulae for the required components of Mel'nikov's vector have the form (everywhere henceforth the integration over t is carried out from $-\infty$ to $+\infty$)

$$M_E(t_0, \theta) = \int g_E(w^*(t + t_0, \theta), t)dt, \quad M_j(t_0, \theta) = \int g_j(w^*(t + t_0, \theta), t)dt$$

(in the general case in the formula for $M_I(t_0,\theta)$ we must subtract $g_I(q, t)$ from $g_I(w^*(t + t_0, \theta), t)$, where I is the first integral of the unperturbed system). Mel'nikov's vector $M(t_0, \theta)$ will have a period in t_0, equal to the period τ of the oscillations of the suspension point.

Thus,

$$-M_E(t_0, \theta) = f_{Ec}(t_0)\cos\theta + f_{Es}(t_0)\sin\theta + f_{E0}(t_0)$$

$$M_j(t_0, \theta) = -f_{jc}(t_0)\cos\theta + f_{js}(t_0)\sin\theta$$

where

$$f_{Ec}(t_0) = \int\dot{\gamma}_\perp(t + t_0)a_x(t)dt \ (c, s;\ x, y), \quad f_{E0}(t_0) = \int\dot{\gamma}_{\|}(t + t_0)a_z(t)dt$$
$$f_{jc}(t_0) = \int\gamma_\perp(t + t_0)a_y(t)dt \ (c, s;\ y, x) \tag{5.1}$$

(an alternative form of these integrals is obtained by making the replacement $t \to t - t_0$). We immediately see from formulae (5.1) that $f'_{jc} = f_{Es}$, $f'_{js} = f_{Ec}$.

To evaluate the integrals (5.1) we will assume that the components of the acceleration are real-analytic functions of time, and we will expand them in Fourier series (everywhere henceforth the summation over k is carried out from $-\infty$ to $+\infty$)

$$a_x(t) = \sum a_{xk}e^{ik\omega t} \ (x, y, z)$$

Here ω is the frequency of the oscillations of the suspension point. The functions $\gamma_{||}(t)$ and $\gamma_\perp(t)$ have an imaginary half-period πi, where

$$\gamma_{\|}(t + \pi i) = \gamma_{\|}(t), \quad \gamma_\perp(t + \pi i) = -\gamma_\perp(t)$$

Hence, using residues, we can easily evaluate the integrals specifying the coefficients for the harmonics in the expansions of the required functions:

$$I_1(\nu) = \int\gamma_\perp(t)e^{i\nu t}dt = \frac{2\pi i\nu}{\mathrm{ch}(\pi\nu/2)}, \quad I_2(\nu) = \int\dot{\gamma}_\perp(t)e^{i\nu t}dt = -i\nu I_1(\nu)$$
$$I_3(\nu) = \int\dot{\gamma}_{\|}(t)e^{i\nu t}dt = \frac{2\pi i\nu^2}{\mathrm{sh}(\pi\nu/2)} \tag{5.2}$$

Thus,

$$f_{Ec}(t_0) = \sum I_2(k\omega)a_{xk}e^{-ik\omega t_0}\,(c, s; x, y), \quad f_{E0}(t_0) = \sum I_3(k\omega)a_{zk}e^{-ik\omega t_0}$$

$$f_{jc}(t_0) = \sum I_1(k\omega)a_{yk}e^{-ik\omega t_0}\,(c, s; y, x)$$

Since all the coefficients (5.2) are non-zero when $v \neq 0$, both components of Mel'nikov's vector are non-constant functions, as at least one of the horizontal components of the acceleration a_x, a_y is a non-constant function of time. If $a_x \equiv \text{const}$ and $a_y \equiv \text{const}$, put the vertical component of the acceleration a_z is non-constant, the function M_j vanishes identically, while the function M_E is non-constant.

For simplicity we will put

$$f_1 = -f_{jc}, \quad f_2 = f_{js}, \quad f_3 = f_{E0}; \quad f = \sqrt{f_1^2 + f_2^2}$$

Then

$$f_{Es} = -f_1', \quad f_{Ec} = f_2'$$

In order to obtain the simple zeroes of Mel'nikov's vector, we rewrite the system considered

$$M_j \equiv f_1(t_0)\cos\theta + f_2(t_0)\sin\theta = 0, \quad -M_E \equiv f_2'(t_0)\cos\theta - f_1'(t_0)\sin\theta + f_3(t_0) = 0 \tag{5.3}$$

in the form

$$(\mathbf{u}\cdot\mathbf{s}) = 0, \quad f_3 - (R^{\pi/2}\mathbf{u}'\cdot\mathbf{s}) = 0$$

where we have introduced the vector $\mathbf{u} = (f_1, f_2)$, which depends on t_0, and its derivative $\mathbf{u}' = (f_1', f_2')$, and also the unit vector $\mathbf{s} = (\cos\theta, \sin\theta)$ and the operation of rotation $R^{\pi/2}$. We will consider the case when $\mathbf{u} \neq 0$, i.e. at least one of the functions f_1 and f_2 does not vanish for the given value of the argument t_0. Then

$$\mathbf{s} = \pm\frac{R^{\pi/2}\mathbf{u}}{|\mathbf{u}|} = \pm\frac{(-f_2, f_1)}{f}, \quad (R^{\pi/2}\mathbf{u}'\cdot\mathbf{s}) = \pm\frac{(\mathbf{u}'\cdot\mathbf{u})}{|\mathbf{u}|} = \pm\frac{(|\mathbf{u}|^2)'}{2|\mathbf{u}|} = \pm|\mathbf{u}|'$$

Thus, the following set of equations follows from system (5.3), provided that $f^2 \neq 0$,

$$\mp f' + f_3 = 0 \tag{5.4}$$

which is equivalent to the single equation

$$(f_1f_1' + f_2f_2')^2 - f^2f_3^2 = 0 \tag{5.5}$$

Note that the mean value of the function f_3 over the period is equal to zero, since $I_3(0) = 0$. Hence, if the functions f_1 and f_2 have no common zeroes, then, on the left-hand side of each of Eq. (5.4) there will be a function, the mean value of which over the period is equal to zero. Then each of the equations has at least two distinct roots t_0. Further, we note that, for each root t_0 of the set of equations (5.4) it is easy to restore (taking into account the sign $\pm$ in (5.4)) the required value of θ and hence the solution (t_0, θ) of the original system (5.3).

We will now prove that the simplicity of the root (t_0, θ) of system (5.3) is equivalent to the simplicity of the root t_0 of the corresponding equation (5.4) (or, which by virtue of the condition $f^2 \neq 0$ is the same, to the simplicity of the root t_0 of Eq. (5.5)).

In fact, the determinant of the Jacobi matrix for system (5.3) at the point (t_0, θ), apart from the sign, is equal to

$$(f_1'\cos\theta + f_2'\sin\theta)^2 + (f_2\cos\theta - f_1\sin\theta)(f_2''\cos\theta - f_1''\sin\theta + f_3')$$

Taking into account the fact that

$$\mathbf{s} = (\cos\theta, \sin\theta) = \pm(-f_2, f_1)/|\mathbf{u}|$$

we can rewrite the Jacobi determinant, apart from a non-zero factor $|\mathbf{u}|^2 = f^2$, as

$$(f_1 f_2' - f_2 f_1')^2 + f^2(f_1 f_1'' + f_2 f_2'' \mp f f_3') = (\mathbf{u} \wedge \mathbf{u}')^2 + |\mathbf{u}|^2((\mathbf{u} \cdot \mathbf{u}'') \mp |\mathbf{u}| f_3') \tag{5.6}$$

where we have used the outer product of the vectors in a plane which is defined in Cartesian coordinates as $(a_x, a_y) \wedge (b_x, b_y) = a_x b_y - a_y b_x$. Since

$$(\mathbf{u} \wedge \mathbf{u}')^2 = |\mathbf{u}|^2 |\mathbf{u}'|^2 - (\mathbf{u} \cdot \mathbf{u}')^2$$

the right-hand side of formula (5.6), apart from a non-zero factor $|\mathbf{u}|^2 = f^2$, can be rewritten as

$$|\mathbf{u}'|^2 - \frac{(\mathbf{u} \cdot \mathbf{u}')^2}{|\mathbf{u}|^2} + (\mathbf{u} \cdot \mathbf{u}'') \mp |\mathbf{u}| f_3' \tag{5.7}$$

Recall that $|\mathbf{u}|' = (\mathbf{u} \cdot \mathbf{u}')/|\mathbf{u}|$, and so

$$|\mathbf{u}|'' = \frac{|\mathbf{u}'|^2 + (\mathbf{u} \cdot \mathbf{u}'')}{|\mathbf{u}|} - \frac{(\mathbf{u} \cdot \mathbf{u}')^2}{|\mathbf{u}|^3}$$

Hence, expression (5.7) vanishes if and only if the derivative of the left-hand side of the corresponding equation (5.4) is equal to zero, i.e. the root t_0 is not simple.

We will now consider the case when $\mathbf{u} = 0$, i.e. the functions f_1 and f_2 vanish for a given value of the argument t_0. Then, the first equation of system (5.3) is automatically satisfied for the given t_0, and, obviously, isolated solutions (t_0, θ) of the system can exist only if $\mathbf{u}' \neq 0$, i.e. at least one of the functions f_1', f_2' does not vanish for the specified t_0. The second equation of (5.3) can then be rewritten in the form

$$|\mathbf{u}'|(\mathbf{p} \cdot \mathbf{s}) - f_3 = 0; \quad \mathbf{p} = (\cos\psi, \sin\psi) = \frac{(-f_2', f_1')}{\sqrt{f_1'^2 + f_2'^2}}$$

Hence,

$$\theta = \psi \pm \arccos \frac{f_3}{\sqrt{f_1'^2 + f_2'^2}} \tag{5.8}$$

The condition for the root (t_0, θ) of system (5.3) to be simple takes the form $f_1' \cos\theta + f_2' \sin\theta \neq 0$, i.e. $\theta - \psi \neq 0 \bmod \pi$.

Thus, the point (t_0, θ) will be a simple root of system (5.3) if the following condition is satisfied

$$|f_3| < |\mathbf{u}'|, \quad \text{i.e.} \quad |f_3| < \sqrt{f_1'^2 + f_2'^2} \tag{5.9}$$

and θ is specified by formula (5.8), where ψ is an angle such that

$$(\cos\psi, \sin\psi) = (-f_2', f_1')/\sqrt{f_1'^2 + f_2'^2} \tag{5.10}$$

Note that the analysis carried out above remains true if the splitting of the complex separatrices (in the complex phase space) is considered. Now f_1, f_2 and f_3 are the functions obtained by extending the real-analytic functions into the complex region, while t_0 and θ are complex quantities and $\mathbf{u} = (f_1', f_2')$, $\mathbf{u}' = (f_1', f_2')$ are complex vectors. Hence, it does not follow from the condition $\mathbf{u} \neq 0$ that $(\mathbf{u} \cdot \mathbf{u}) \equiv f^2 \neq 0$, and it does not follow from the condition $\mathbf{u}' \neq 0$ that $(\mathbf{u}' \cdot \mathbf{u}') \equiv f_1'^2 + f_2'^2 \neq 0$. However, according to the first equation of (5.3) we obtain $(-f_2, f_1) = k(\cos\theta, \sin\theta)$ with a constant k, whence $(\mathbf{u} \cdot \mathbf{u}) = k^2$, and therefore $\mathbf{u} = 0$ as soon as $(\mathbf{u} \cdot \mathbf{u}) = 0$. Further, condition (5.9) is now replaced by the condition $f_1'^2 + f_2'^2 \neq f_3^2$.

Suppose now that $\mathbf{u} = 0$ and $(\mathbf{u}' \cdot \mathbf{u}') = 0$ for the given value of t_0, whereas $\mathbf{u}' \neq 0$. Then $f_1' = jf_2'$, where $j \in \{+i, -i\}$ and i is the square root of -1. The second equation of (5.3) can be rewritten in the form

$$\exp(-j\theta) = -f_3/f_2' \tag{5.11}$$

Note that

$$f'_1\cos\theta + f'_2\sin\theta = j(f'_2\cos\theta - f'_1\sin\theta) = -jf_3$$

Hence, with the condition $f_3 \neq 0$ system (5.3) has the root (t_0, θ), where θ is found from Eq. (5.11), and this root is simple.

In the next section we will show that each unperturbed complex separatrix $W_0^{\pm}$ is occupied not only by doubly asymptotic trajectories of "non-isotropic" solutions $w = w^*(t + t_0, \theta)$, but also contains two different trajectories of "isotropic" solutions, for which $\gamma_x^2 + \gamma_y^2 = 0$ and $\dot\gamma_x^2 + \dot\gamma_y^2 = 0$. However, these trajectories turn out to be asymptotic, but not doubly asymptotic, i.e. they only pertain to one of the two separatrices.

The results obtained are summed up in the following theorem.

Theorem 4. *The separatrices are split for any periodic motion of the suspension point with variable acceleration. All the simple zeroes (t_0, θ) of the two-dimensional Mel'nikov vector are described as follows in terms of one-dimensional equations.*

1°. To each simple root t_0 of Eq. (5.4) (for one of the two signs ∓); such that $f^2 \neq 0$ at the point t_0, there corresponds a simple zero (t_0, θ) of Mel'nikov's vector where the angle θ^- is found from the condition

$$(\cos\theta, \sin\theta) = \pm(-f_2, f_1)/f$$

2°. If the conditions $f_1 = 0$ and $f_2 = 0$ and (5.9) are satisfied at the point t_0, there will be a pair of simple zeroes (t_0, θ) of Mel'nikov's vector, where the angles θ are found from formulae (5.10) and (5.8).

If we consider complex separatrices, then assertion 1° remains true, and the conditions of assertion 2° will be slightly modified, as described below, and the new assertion 3° will appear.

$2°_1$. If the following conditions are satisfied at the point t_0

$$f_1 = 0, \quad f_2 = 0, \quad f_1'^2 + f_2'^2 \neq 0, \quad f_1'^2 + f_2'^2 \neq f_3^2$$

then there will be a pair of simple zeros (t_0, θ) of Mel'nikov's vector, where the angles θ are found from formulae (5.10) and (5.8).

3°. If the following conditions are satisfied at the point t_0

$$f_1 = 0, \quad f_2 = 0, \quad f_3 \neq 0, \quad f_1'^2 + f_2'^2 = 0$$

while f_1' and f_2' are non-zero (so that $f_1' = jf_2'$, where $j^2 = -1$), there will be a simple zero (t_0, θ) of Mel'nikov's vector, where the angle θ is found from formula (5.11).

6. The non-integrability of the system of equations of the perturbed spherical pendulum

Suppose the τ-periodic vector-function $\mathbf{a}(t)$ is holomorphic everywhere in the domain $V \subset \mathbb{C}$, apart from singularities t_k, and $\Gamma = \Gamma_0$: [0, 1] → V is a certain contour, such that $\Gamma(1) - \Gamma(0) = \tau$. We will choose closed contours $\Gamma_k \subset V$ ($k \neq 0$), passing through the point $t_0 = \Gamma_0(0)$ such that each of them is contractible in the domain V and surrounds exactly one singularity $t_k \in V$ of the function f. We will introduce the two-dimensional vectors

$$\mathbf{F}^{(\pm, k)} = \mathrm{Res}_{t = t_k}[(a_x(t), a_y(t))e^{\pm t}] = (F_x^{(\pm, k)}, F_y^{(\pm, k)})$$

$$F_x^{(\pm, k)} = \mathrm{Res}_{t = t_k}(a_x(t)e^{\pm t}) \ (x, y)$$

We will associate with each simple zero $t_{0,m}$, θ_m of the Mel'nikov vector the two-dimensional vector

$$\mathbf{G}^{(m)} = (\cos\theta_m, \sin\theta_m)$$

and the unperturbed doubly asymptotic solution

$$\gamma_m : w = w^*(t + t_{0,m}, \theta_m)$$

The following non-integrability theorem directly supplements Theorem 4.

Theorem 5. *If, for each of the two indices ± among the vectors $F^{(\pm,k)}$, $G^{(m)}$ there is a pair of non-collinear vectors, the perturbed system is non-integrable in any neighbourhood of the closed set*

$$\bigcup_m \gamma_m \cup \left(\{q\} \times \bigcup_k \Gamma_k\right) \subset D$$

for all sufficiently small $\varepsilon \neq 0$.

The proof is based on checking of the conditions of Theorem 3. For this purpose we consider the phase flow ϕ^t of an autonomous system – the unperturbed spherical pendulum, and we construct two-dimensional linearizing coordinates $u^{\pm}$ on $W_0^{\pm}$, in which the phase flow takes a linear form. We will first investigate the phase flow $\overset{\circ}{\phi}{}^t$ for the plane pendulum. The separatrices $\overset{\circ}{W}{}_0^{\pm}$ of its unstable equilibrium position q are one-dimensional curves, such that $\overset{\circ}{W}{}_0^{\pm} \setminus \{q\}$ consists of two connected components, which are the trajectories of the doubly-asymptotic solutions s_+: $w = w^*(t, 0)$ and s_-: $w = w^*(t, \pi)$. The corresponding characteristic exponent of the phase flow on $\overset{\circ}{W}{}_0^{\pm}$ at the fixed point q is equal to ±1.

We will introduce the coordinate $\rho^{\pm}$ on $\overset{\circ}{W}{}_0^{\pm} = \{q\} \cup s_+ \cup s_-$, assuming $\rho^{\pm} = 0$ at the point q, $\rho^{\pm} = \exp(\mp t)$ at the point $w = w^*(t, 0) \in s_+$ and $\rho^{\pm} = -\exp(\mp t)$ at the point $w = w^*(t, \pi) \in s_-$ (the manifold $\overset{\circ}{W}{}_0^{\pm}$ is obtained by "sticking" the point q to those ends of the trajectories s_+ and s_- which correspond to the limit transition $t \to \pm\infty$). Then $\rho^{\pm}$ is an analytic linearizing coordinate on $\overset{\circ}{W}{}_0^{\pm}$, i.e. the phase flow on $\overset{\circ}{W}{}_0^{\pm}$, being written in this coordinate, takes the linear form $\phi_0^t(\rho^{\pm}) = \exp(\mp t)\rho^{\pm}$, and the corresponding vector field of the phase flow has the form $\mp\rho^{\pm}$.

Suppose l is a straight line tangent to the configuration space

$$S^1 = \{\boldsymbol{\gamma}: \gamma_x^2 + \gamma_z^2 = 1, \gamma_y = 0\}$$

of the plane pendulum at the point $O = (0, 0, 1)$. Then the straight line l is parallel to the x axis. Identifying the variable $\rho^{\pm}$ with the x coordinate of a point on l, we will assume that the linearizing coordinate on $\overset{\circ}{W}{}_0^{\pm}$ ranges over the straight line l.

The separatrices of the spherical pendulum $W_0^{\pm}$ in the real region are obtained from the separatrices of the plane pendulum $\overset{\circ}{W}{}_0^{\pm}$ by means of the transformations $T^{\theta} = (R_z^{\theta}, R_z^{\theta})$ of the phase space $\mathbb{R}^6\{(\gamma, \dot{\gamma})\}$, which are generated by the rotations R_z^{θ} of the configuration space around the vertical z axis. By applying rotations R_z^{θ} to the one-dimensional linearizing coordinate $x \in l$ on $\overset{\circ}{W}{}_0^{\pm}$, we obtain two-dimensional coordinates $u^{\pm}$ on $W_0^{\pm}$ in which the phase flow takes the linear form

$$\phi^t(u^{\pm}) = \exp(\mp t)u^{\pm}$$

Hence, the coordinates $u^{\pm} = (x, y)$ correspond to the point $w = w^*(t, \theta)$ on $W_0^{\pm}$, where

$$x = \rho\cos\theta, \quad y = \rho\sin\theta, \quad \rho = \exp(\mp t)$$

Thus, we have constructed linearizing coordinates on $W_0^{\pm}$, which range over the plane $P\{(x, y)\}$, identified with the coordinate plane xy. We will show below that these linearizing coordinates are analytic, i.e. the mapping $\psi^{\pm} : P \to W_0^{\pm}$, which makes the set of linearizing coordinates correspond to a point on the separatrice, will be analytic. A similar construction for the separatrix $W_0^{\pm}$ in the complex region only gives part of this separatrix, occupied by trajectories of the non-isotropic solutions and parametrized by the linearizing coordinates $u^{\pm} = (x, y)$, that ranges over the domain of non-isotropicity $\{(x, y): x^2 + y^2 = \rho^2 \neq 0\}$. It can be seen that when using these linearizing coordinates the following element will correspond to the solution γ_m

$$z_m^{\pm} = \exp(\mp t_{0,m})\mathbf{G}^{(m)}$$

Close to the point $O = (0, 0, 1)$ of the configuration space $S^2 = \{\gamma: |\gamma| = 1\}$ of the spherical pendulum, it is convenient to use $\upsilon = (x, y)^{\tau} = (\gamma_x, \gamma_y)^{\tau}$ as local coordinates (the operation of transposition is used, since the vectors will be regarded as column vectors). Then $w_0 = (x, \dot{x}, y, \dot{y})^{\tau}$ are the corresponding local coordinates in the phase space TS^2 near q. The kinetic and potential energies are expanded in series

$$T = \frac{1}{2}(\dot{\boldsymbol{\gamma}}, \dot{\boldsymbol{\gamma}}) = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + \ldots, \quad U = z + \varepsilon(\mathbf{a}, \boldsymbol{\gamma}) = \varepsilon a_x x + \varepsilon a_y y - \frac{1}{2}(x^2 + y^2) + \ldots$$

with omitted terms $O_3(x, y)$ and $O(\varepsilon)O_2(x, y)$, where $O_k(x, y)$ denotes terms of the order k in x and y. Hence, the linearized equations of the unperturbed spherical pendulum at the point q have the form $\ddot{\upsilon} = \upsilon$, or

$$\dot{w}_{\circ} = \Lambda w_{\circ}; \quad \Lambda = \begin{pmatrix} \Lambda_0 & 0 \\ 0 & \Lambda_0 \end{pmatrix}, \quad \Lambda_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

while the principal terms, which occur on the right-hand side of the perturbed system, constitute a vector $\varepsilon Y(t)$, where

$$Y(t) = X_1(q, t) = (0, -a_x(t), 0, -a_y(t))^{\top}$$

It can be seen that the invariant subspaces $L^{\pm}$ of the linearized system are specified by the relations $\dot{\upsilon} = \mp\upsilon$ (this can also be obtained by considering the asymptotic behaviour of the solutions $w = w^*(t, \theta)$ as $t \to \pm\infty$. Further, the projections $\pi^{\pm} : \mathbb{C} \to L^{\pm}$ are specified by the formulae

$$\pi^{\pm}(x, \dot{x}, y, \dot{y}) = (\alpha_x^{\pm}, \mp\alpha_x^{\pm}, \alpha_y^{\pm}, \mp\alpha_y^{\pm}); \quad \alpha_x^{\pm} = (x \pm \dot{x})/2 \ (x, y)$$

Suppose $l^{\pm}: P \to L^{\pm}$ is an isomorphism of the linear spaces, coinciding with the differential $d_0\psi^{\pm}$ of the mapping $\psi\pm : P \to W_0^{\pm}$ at zero. Then

$$\chi^{\pm} = \psi^{\pm} \circ (l^{\pm})^{-1} : L^{\pm} \to W_0^{\pm}$$

will be the required mapping, which satisfies the condition $d_0\chi^{\pm} = \mathrm{id}$ and which specifies the linearizing coordinates on $W_0^{\pm}$, ranging over $L^{\pm}$. Since

$$\gamma_{\perp}(t) \sim \pm 4\exp(\mp t) \quad \text{as} \quad t \to \pm\infty$$

we have

$$l^{\pm}(x, y) = d_0\psi^{\pm}(x, y) = 4(\pm x, -x, \pm y, -y)$$

The requirement that the linearizing coordinates on the separatrice $W_0^{\pm}$ should range over the tangent subspace $L^{\pm} = T_q W_0^{\pm}$ was convenient for formulating Theorem 3. However, for practical use of the theorem it is necessary to introduce convenient coordinates on the linear space $L^{\pm}$ itself and to use them as the linearizing coordinates on $W_0^{\pm}$. We will apply the isomorphism $l^{\pm}: P \to L^{\pm}$ so as to introduce coordinates on $L^{\pm}$, which ranges over the plane P. Hence, we return from the linearizing coordinates, specified by the mapping $\chi^{\pm} : L^{\pm} \to W_0^{\pm}$ to the initial linearizing coordinates, specified by the mapping $\psi\pm : P \to W_0^{\pm}$.

The point $(\alpha_x^{\pm}, \mp\alpha_x^{\pm}, \alpha_y^{\pm}, \mp\alpha_y^{\pm}) \in L^{\pm}$ will obviously have coordinates $\pm(\alpha_x^{\pm}, \alpha_y^{\pm})/4 \in P$. Using the formulae for the vector $X_1(q, t)$ and the projections $\pi^{\pm} : \mathbb{C}^4 \to L^{\pm}$, we obtain that the $L^{\pm}$-valued function $f^{\pm}(t) = \pi^{\pm} \exp(\pm t)X_1(q, t)$ in these coordinates takes the form $-\exp(\pm t)(a_x(t), a_y(t))/8$ (here $\Lambda\pm = \mp\mathrm{id}$ is the scalar operator by virtue of the symmetry of the problem).

Hence

$$\mathrm{Res}_{t = t_k} f^{\pm} = -\mathbf{F}^{(\pm, k)}/8$$

We will now verify that the complex linearizing coordinates (x, y) on the complex separatrix $W_0^{\pm}$ are analytic in the vicinity of the origin of coordinates, and we will give a description of the "isotropic" part of the separatrix. If the point $w = \overset{\circ}{\psi}{}^{\pm}(\rho)$ on $\overset{\circ}{W}{}_0^{\pm}$ corresponds to the coordinate $\rho^{\pm} = \rho$, the point $\psi^{\pm}(x, y) = T^{\theta}(w)$ on $W_0^{\pm}$ will correspond to the coordinates $(x, y) = (\rho \cos \theta, \rho \sin \theta)$. By virtue of the property of symmetry for the coordinates of the point $w = \overset{\circ}{\psi}{}^{\pm}(\rho)$ we have the expressions

$$\gamma_x = \pm 4\rho + \rho f(\rho^2), \quad \gamma_y = 0, \quad \gamma_z = 1 + \rho^2 g(\rho^2)$$

$$\dot{\gamma}_x = -4\rho + \rho f_1(\rho^2), \quad \dot{\gamma}_y = 0, \quad \dot{\gamma}_z = \rho^2 g_1(\rho^2)$$

where $f(\rho^2)$, $g(\rho^2)$, $f_1(\rho^2)$, $g_1(\rho^2)$ are meromorphic functions, having a unique singularity at the point $\rho^2 = -1$, which corresponds to the singularities of the functions $\gamma_{\perp}(t)$, $\gamma_{||}(t)$. Hence, the coordinates of the point $T^{\theta}(w)$ are expressed by the formulae

$$\gamma_x = \pm 4x + xf(\rho^2), \quad \gamma_y = \pm 4y + yf(\rho^2), \quad \gamma_z = 1 + \rho^2 g(\rho^2)$$

$$\dot{\gamma}_x = -4x + xf_1(\rho^2), \quad \dot{\gamma}_y = -4y + yf_1(\rho^2), \quad \dot{\gamma}_z = \rho^2 g_1(\rho^2); \quad \rho^2 = x^2 + y^2$$

which specify the mapping $\psi^{\pm}$ in the domain of non-isotropicity. Obviously the mapping is extendable to the set of isotropic coordinates (x, y), $x^2 + y^2 = 0$, and takes the form there

$$\gamma_x = \pm 4x, \quad \gamma_y = \pm 4y, \quad \gamma_z = 1, \quad \dot{\gamma}_x = -4x, \quad \dot{\gamma}_y = -4y, \quad \dot{\gamma}_z = 0$$

The last formulae specify the trajectories of the isotropic solutions, while the motion along these trajectories is defined by the dynamics of the coordinates

$$(x, y) = \exp(\mp t)(x_0, y_0)$$

These formulae of the isotropic solutions can also be easily obtained directly from the equation of motion of the spherical pendulum

$$\ddot{\boldsymbol{\gamma}} = (\gamma_z - (\dot{\boldsymbol{\gamma}} \cdot \dot{\boldsymbol{\gamma}}))\boldsymbol{\gamma} + \mathbf{g}$$

where $\mathbf{g} = (0, 0, -1)$ is the acceleration due to gravity.

Thus, the mapping $\psi^{\pm}$ is meromorphic everywhere on $\mathbb{C}^2\{(x, y)\}$ and has singularities at the points where $x^2 + y^2 = -1$. On each separatrix $W_0^{\pm}$ there are exactly two trajectories of isotropic solutions (corresponding to the two cases $x = \pm iy$, where i is the square root of -1). These solutions have no limit as $t \to \mp\infty$, i.e. they do not belong to the second separatrix $W_0^{\mp}$.

The condition of Theorem 5 denotes that the linear hull of the set $K^{\pm}$ coincides with the whole plane $\mathbb{C}^2$ and, according to Definition 1′, $J^{\pm}$ and $K^{\pm}$ are in general position, i.e. the conditions of Theorem 3 are satisfied.

Example. For horizontal sinusoidal oscillations of the suspension point,

$$a_x(t) = \cos\omega t, \quad a_y(t) = a_z(t) = 0$$

Then

$$-M_E(t_0, \theta) = 2\pi\frac{\omega^2 \cos\omega t_0 \cos\theta}{\operatorname{ch}(\pi\omega/2)}, \quad M_j(t_0, \theta) = 2\pi\frac{\omega \sin\omega t_0 \sin\theta}{\operatorname{ch}(\pi\omega/2)}$$

Hence Mel'nikov's vector has four zeroes (on a rectangle of periods)

$$\omega t_0 = \pm\frac{\pi}{2} \bmod 2\pi, \quad \theta = 0 \bmod 2\pi; \quad \omega t_0 = 0 \bmod 2\pi, \quad \theta = \pm\frac{\pi}{2} \bmod 2\pi$$

and they are all simple. The first two zeroes correspond to a pair of transversal homoclinic solutions, lying in the xz plane. These are homoclinic solutions for the plane pendulum, oscillating in the given plane. The two other zeroes correspond to a pair of transversal homoclinic solutions, which are close to the doubly asymptotic solutions of the unperturbed problem, which lie in the *yz* plane, perpendicular to the direction of the oscillations of the suspension point. Theorem 3 obviously gurantees the non-integrability of the perturbed system in the real domain.

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